ON COMPRESSION SHOCKS IN THREE-DIMENSIONAL FLOWS WITH A DEGENERATE HODOGRAPH

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Steady and unsteady three-dimensional flows with degenerate hodographs, not belonging to the class of simple waves, have been studied in [1 to 3]. There the flow region in the phase space $x_1 x_2 x_3 t$ corresponded in the hodograph space $u_1 u_2 u_3$ either to a certain surface – for the case of double waves – or to a certain three-dimensional region – for the case of triple waves (u_i are the components of the velocity vector).

For polytropic gas under the assumption of isentropic and potential character of flows considered in [1 to 3], systems of equations describing the corresponding classes of flow in the hodograph space were deduced.

Below we investigate flows behind three-dimensional shock waves when it is assumed that the surface of discontinuity is represented by a certain curve in the hodograph space, whilst the flow behind the shock wave belongs to the class of double waves. Essentially, we consider only shock (detonation) waves of constant intensity, since the flow behind the wave front is assumed isentropic. For the system of equations describing the double waves along certain lines in the plane of the independent components of velocity, a Cauchy problem is formulated. The system of equations under consideration turns out to be elliptic behind the front of the shock waves and hyperbolic behind normal detonation waves. It appears that in the steady case behind the surface of a strong discontinuity the velocity of the sound as a function of the velocity components is the same as in the case of a conical self-similar flow. This gives the possibility of obtaining certain exact solutions for steady three-dimensional flow past certain bodies of special shape in the presence of shock fronts.

Flows behind the surface of a strong discontinuity in the class of plane unsteady double waves have been studied also in [4 and 5].(*)

1. The system of equations describing unsteady three-dimensional double waves can be written down in the following form [2 and 3]

*) A.F. Sidorov, Nekotorye tochnye resheniia nestatsionarnoi mnogomernoi gazovoi dinamiki (Certain exact solutions of unsteady multi-dimensional gas dynamics). Dissertation, Institute of Hydrodynamics SO Akad.Nauk SSSR,1963.

$$R_{11}\Psi_{22} - 2R_{12}\Psi_{12} + R_{22}\Psi_{11} = 0$$
$$R_{11}\Lambda_{22} - 2R_{12}\Lambda_{12} + R_{22}\Lambda_{11} = 0 \qquad (1.1)$$
$$R_{11}X_{22} - 2R_{12}X_{12} + R_{22}X_{11} = 0$$

Here

$$R_{ik} = - [\Lambda_i - \Psi\Psi_i - u_i)(\Lambda_k - \Psi\Psi_k - u_k) + \theta(\delta_{ik} + \Psi_i\Psi_k) \qquad (1.2)$$
$$(i,\ k = 1, 2,\ \delta_{ik} = 0 \ \text{for}\ i \neq k,\ \delta_{ii} = 1)$$
$$\Psi_{ik} = \frac{\partial^2\Psi}{\partial u_i \partial u_k},\ \Lambda_{ik} = \frac{\partial^2\Lambda}{\partial u_i \partial u_k},\ X_{ik} = \frac{\partial^2 X}{\partial u_i \partial u_k},\ \Psi_i = \frac{\partial\Psi}{\partial u_i},\ \Lambda_i = \frac{\partial\Lambda}{\partial u_i},\ X_i = \frac{\partial X}{\partial u_i}$$
$$\theta = \frac{1}{\varkappa}\left[\Lambda - \frac{1}{2}(u_1^2 + u_2^2 + \Psi^2)\right], \qquad \varkappa = \frac{1}{\gamma - 1} \qquad (1.3)$$
$$u_3 = \Psi(u_1, u_2) \qquad (1.4)$$

The flow is assumed to be isentropic; the equation of state will be taken in the form

$$p = a^2\rho^\gamma, \quad \theta = c^2, \qquad a = \text{const}$$

Here p is the pressure, γ is the adiabatic index, ρ is the density and c is the sound velocity.

The velocity components u_1 and u_2 are assumed to be functionally independent. After solution of the system of equations (1.1) for the functions Ψ, Λ and X, the flow in the phase space $x_1 x_2 x_3 t$ is found from the relations

$$\frac{\partial\nabla}{\partial u_i} = x_i + \Psi_i x_3 \ (i = 1, 2), \quad \nabla = \Lambda(u_1, u_2)\, t + X(u_1, u_2) \qquad (1.5)$$

Here ∇ is a "distribution" function, connected with the velocity potential φ by the relation

$$\nabla = \sum_k u_k x_k - \varphi \qquad (1.6)$$

Suppose that a three-dimensional shock wave, so far of arbitrary shape, propagates into stationary uniform gas ($u_i = 0$, $i = 1, 2, 3$ and $\rho = \text{const}$) with a constant normal velocity D. From the Hugoniot conditions for this case it follows that the jumps in entropy S in the modulus of the velocity $|\mathbf{u}|$ and in the sound velocity c are constant along the surface of discontinuity L. Suppose that

$$u_1^2 + u_2^2 + \Psi^2 = A^2 = \text{const} \quad \text{on } L \qquad (1.7)$$

Let us consider the following problem. What degree of arbitrariness of solution pertains to the equations of hydrodynamics and what properties characterize the surface of discontinuity, if the shock wave corresponds to a certain curve l on the sphere (1.7), whilst the flow behind the wave front belongs to the class of potential double waves. At the same time we shall explore the question of formulation of problems for the system of equations corresponding to double waves. Let us assign the equation of the projection of the curve l on the $u_1 u_2$ plane in the form

$$u_2 = f(u_1) \qquad (1.8)$$

This can always be done without loss of generality. The curve l is then defined by Equations (1.7) and (1.8), whilst the motion of the surface of discontinuity is determined by the relations

$$\Lambda_i t + X_i = x_i + \Psi_i x_3 \qquad (i = 1, 2) \tag{1.9}$$

following from (I.5), where instead of u_2 we have substituted its expression in terms of u_1, according to (1.8).

Let us consider an srbitrary instant of time $t = t_0$ and find expressions for two linearly independent vectors τ_1 and τ_2 lying in the tangent plane to the surface of discontinuity, the expression for the normal $\mathbf{n}$ to the surface of discontinuity and the normal velocity of its motion D.

From (1.9), assuming that the surface is given by the parametric expressions $x_i = \Phi_i(s, v)$ $(s = u_1, \ v = x_3)$, we obtain

$$\tau_1 = (\tau_{1s}, \tau_{2s}, 0), \quad \tau_2 = (-\Psi_1, -\Psi_2, 1) \tag{1.10}$$

$$\tau_{is} = t_0(\Lambda_{i1} + \Lambda_{i2}f') + (X_{i1} + X_{i2}f') - (\Psi_{i1} + \Psi_{i2}f')\, x_3 \quad (i = 1, 2) \tag{1.11}$$

For the normal $\mathbf{n}$, since the gas is at rest in front of the shock wave, we have

$$\mathbf{n} = \frac{1}{|\mathbf{u}|}(u_1, u_2, \Psi) \tag{1.12}$$

Finally, by writing Equations (1.9) in the form

$$F_i(x_1, x_2, x_3, t, s) = 0 \quad (i = 1, 2),$$

we can find the normal velocity D from the relation

$$D = \left|\frac{\partial(F_1, F_2)}{\partial(t, s)}\right| \left[\sum_k \left(\frac{\partial(F_1, F_2)}{\partial(x_k, s)}\right)^2\right]^{-1/2} \tag{1.13}$$

The Hugoniot conditions together with the conditions

$$\mathbf{n}\cdot\tau_i = 0 \qquad (i = 1, 2) \tag{1.14}$$

and Equation (1.13), where D = const the same as in the Hugoniot conditions, give all the relations which must be fulfilled on the surface of discontinuity for the given case. Let us proceed to their analysis. Conditions (1.14) give

$$\tau_{1s}u_1 + \tau_{2s}u_2 = 0, \qquad \Psi - u_1\Psi_1 - u_2\Psi_2 = 0 \tag{1.15}$$

Let us substitute in the first of these equations Expressions (1.10) for τ_{is}; then, making use of the fact that t_0 and x_3 are arbitrary, we find that the equation under consideration is equivalent to the three equations

$$\begin{aligned} u_1(\Psi_{11} + \Psi_{12}f') + u_2(\Psi_{21} + \Psi_{22}f') &= 0 \\ u_1(\Lambda_{11} + \Lambda_{12}f') + u_2(\Lambda_{21} + \Lambda_{22}f') &= 0 \\ u_1(X_{11} + X_{12}f') + u_2(X_{21} + X_{22}f') &= 0 \end{aligned} \tag{1.16}$$

valid along the curve $u_2 = f(u_1)$.

With the help of (1.15), the relation (1.13) is easily reduced to the form

$$D = \frac{|\Lambda_1 u_1 + \Lambda_2 u_2|}{\sqrt{u_1^2 + u_2^2 + \Psi^2}} \qquad (1.17)$$

Integrating equations (1.16) along $u_2 = f(u_2)$, we obtain the integrals

$$\Psi = u_1\Psi_1 + u_2\Psi_2 + K_1, \quad \Lambda = u_1\Lambda_1 + u_2\Lambda_2 + K_2, \quad X = u_1X_1 + u_2X_2 + K_3$$
$$(K_i = \text{const}) \qquad (1.18)$$

Accordingly, from equations (1.3), (1.7), (1.17) and (1.18) we obtain on the curve $u_2 = f(u_1)$ the following initial data for Cauchy's problem:

for the function Ψ

$$u_1^2 + f^2 + \Psi^2 = A^2, \quad \Psi = u_1\Psi_1 + u_2\Psi_2, \quad K_1 = 0 \qquad (1.19)$$

for the function Λ

$$u_1\Lambda_1 + u_2\Lambda_2 = \pm DA, \qquad K_2 = B^2 \mp DA$$
$$\Lambda = B^2 = \text{const}, \qquad B^2 = \varkappa\theta + {}^1/_2 A^2 \qquad (1.20)$$

For the function X we have only one equation (1.18) and consequently have available in its determination a unifunctional degree of arbitrariness. It follows from Formula (1.9) that the constant K_3 in Equation (1.18) for X is immaterial and it may be set equal to zero.

Equations (1.9) may be put in parametric form

$$\mathbf{X} = \mathbf{K}(s) + v\mathbf{P}(s), \qquad \mathbf{X} = (x_1, x_2, x_3) \quad (s \text{ and } v \text{ are parameters}) \qquad (1.21)$$

$$\mathbf{K}(s) = (X_1 + t_0\Lambda_1, \ X_2 + t_0\Lambda_2, 0), \qquad \mathbf{P}(s) = (-\Psi_1, \ -\Psi_2, 1) \qquad (1.22)$$

From (1.21) it follows that the surface of discontinuity under consideration is a ruled surface. We shall show that it is a developable surface. Setting down the condition for developability, we shall have

$$\mathbf{K}'\mathbf{P}\mathbf{P}' = \begin{vmatrix} -\Psi_1 & -\Psi_2 & 1 \\ -(\Psi_{11} + \Psi_{12}f') & -(\Psi_{21} + \Psi_{22}f') & 0 \\ X_{11} + X_{12}f' + t_0(\Lambda_{11} + \Lambda_{12}f') & X_{21} + X_{22}f' + t_0(\Lambda_{21} + \Lambda_{22}f') & 0 \end{vmatrix} = 0 \qquad (1.23)$$

by virtue of condition (1.16), i.e. the surface of discontinuity is in fact developable.

Let us consider the fixed instant of time $t = 0$ (this can always be achieved by a displacement in respect to time) and the curve obtained as the section of the surface of discontinuity by the plane $x_3 = 0$. From (1.9) it follows that the equations of this curve in the $x_1 x_2$ plane are

$$x_i = X_i \qquad (i = 1, 2) \qquad (1.24)$$

where $u_2 = f(u_1)$. Accordingly, the assignment on the curve (1.8) of a second functional dependence for the function X in the form $\Phi(X_1, X_2) = 0$ corresponds to assignment of a certain director curve $\Phi(x_1, x_2) = 0$ in the $x_1 x_2$ plane for the developable surface under consideration. In this connection the assignment of the dependence (1.8) determines the position of the generators of this surface, i.e. altogether in the determination of the surface of discontinuity we have available the degree of arbitrariness in two functions of a single independent variable.

From the foregoing we draw the following conclusion.

T h e o r e m 1.1 . If in hodograph space a certain curve corresponds to the surface of a strong discontinuity, and the flow behind the discontinuity belongs to the class of potential double waves, then this surface is a developable surface at any instant of time $t = t_0$ and Cauchy's problem for the system (1.1) can be formulated for any surface developable in $x_1 x_2 x_3$ space when $t = t_0$.

We notice that after assigning the shape of the surface of discontinuity at a certain instant of time $t = t_0$, the position of the surface of discontinuity at any other instant of time t is determined immediately by (1.9), in which Λ_i, X_i and Ψ_i are found from the Cauchy initial data on the curve $u_2 = f(u_1)$.

2. We note that the coefficients of the equations for the functions Ψ, Λ and X are identical and the type of system (1.1) is determined by the sign of Expression

$$R = R_{12}^2 - R_{11}R_{22} \tag{2.1}$$

When $R > 0$, the system of equations (1.1) is of hyperbolic type when $R < 0$ it is elliptic.

Making use of (1.2), let us write the expression for R in the form

$$R = \theta \{(\Lambda_1 - \Psi\Psi_1 - u_1)^2 + (\Lambda_2 - \Psi\Psi_2 - u_2)^2 + [\Psi_1(\Lambda_2 - \Psi\Psi_2 - u_2) - \Psi_2(\Lambda_1 - \Psi\Psi_1 - u_1)]^2 - \theta(1 + \Psi_1^2 + \Psi_2^2)]\} \tag{2.2}$$

Let us find the value of R on the curve $u_2 = f(u_1)$. From Formulas (1.20) for Λ_1 and Λ_2 we obtain

$$\Lambda_1 = -\frac{Lf'}{f - f'u_1}, \qquad \Lambda_2 = \frac{L}{f - f'u_1} \qquad (L = AD) \tag{2.3}$$

(the case $L = -AD$ corresponds to the propagation of a shock wave into a certain field changing in a special manner and it will not be considered in the present paper). For Ψ_1 and Ψ_2 from (1.19) we have

$$\Psi_1 = -\frac{f'\Psi^2 + f'f^2 + u_1 f}{\Psi(f - f'u_1)}, \qquad \Psi_2 = \frac{\Psi^2 + u_1^2 + u_1 ff'}{\Psi(f - f'u_1)} \tag{2.4}$$

where Ψ is determined by the relation (1.7).

Substituting the expressions for the derivatives from (2.3) and (2.4) and performing the necessary transformations, we finally obtain for R

$$R = \frac{\theta[\Psi^2(1 + f'^2) + (u_1 + ff')^2]}{\Psi^2(f - f'u_1)^2}[(L - A^2)^2 - \theta A^2] \tag{2.5}$$

We notice that the case $f - f'u_1 = 0$ is of no interest and leads to the relation $u_1 = 0$. In fact, setting $u_2 = pu_1$ (p = const) we obtain from (1.19)

$$u_1 + p^2u_1 + \Psi(\Psi_1 + \Psi_2 p) = 0, \qquad \Psi = u_1(\Psi_1 + p\Psi_2)$$

Hence

$$u_1(1 + p^2 + (\Psi_1 + \Psi_2 p)^2) = 0, \quad \text{i.e. } u_1 = 0$$

The sign of R coincides with the sign of Expression

$$T = (L - A^2)^2 - \theta A^2$$

Recalling the meaning of the notation introduced, let us write T in the form

$$T = |\mathbf{u}|^2 (D - |\mathbf{u}| - c)(D - |\mathbf{u}| + c) \tag{2.6}$$

At the front of an ordinary shock wave, propagating in undisturbed gas, the inequalities

$$|\mathbf{u}| + c > D \qquad D > |\mathbf{u}| \tag{2.7}$$

are fulfilled (disturbances behind the front overtake the shock wave); consequently, $R < 0$, and the system of equations (1.1) in this case is of elliptic type.

If the surface of discontinuity L is the front of a normal detonation, at which the Chapman-Jouguet condition

$$|\mathbf{u}| + c \doteq D \tag{2.8}$$

is fulfilled, then from (2.6) it follows that $T = R = 0$, i.e. the curve $u_2 = f(u_1)$ is a curve indicating parabolic character of the system (1.1), whilst to determine the type of the system (1.1) behind the detonation front a supplementary investigation is needed.

We notice that the curve $u_2 = f(u_1)$ is not a characteristic of the system (1.1). In fact, using the relations (2.3) and (2.4) we have

$$\begin{aligned} R_{11}du_1^2 + 2R_{12}du_1du_2 + R_{22}\,du_2^2 &= du_1^2\,[R_{11} + 2R_{12}f' + R_{22}f'^2) = \\ &= du_1^2\theta\,[1 + f'^2 + (\Psi_1 + \Psi_2 f')^2] > 0 \quad (\text{for } u_1 \neq \text{const}) \end{aligned} \tag{2.9}$$

Let us consider the arbitrary point $M(u_1, u_2, \Psi)$ lying on the curve l. The curve l is the line of intersection of the sphere (1.7) and the surface $u_3 = \Psi(u_1, u_2)$, into which the region of flow behind the surface of discontinuity is transformed in the hodograph space. Behind the front of a normal detonation wave the pressure and the modulus of the velocity decrease, therefore it is sufficient to consider the part of the surface $u_3 = \Psi(u_1, u_2)$ lying inside the sphere (1.7), and study the sign of R inside this sphere.

We shall show first of all that at the point M the surface (1.4) cannot touch the sphere (1.7). Vectors directed along the normals to the surface (1.4) and the sphere (1.7) at the point M can be written, respectively, in the form

$$\mathbf{n}_\psi = (-\Psi_1, \ -\Psi_2, 1), \qquad \mathbf{n}_s = (u_1, u_2, \Psi) \tag{2.10}$$

In the case of tangency the vectors $\mathbf{n}_\psi$ and $\mathbf{n}_s$ are proportional

$$\mathbf{n}_\psi = c\mathbf{n}_s, \quad c \neq 0, \quad \text{or} \quad -\Psi_1 = cu_1, \quad -\Psi_2 = cu_1, \quad 1 = c\Psi \tag{2.11}$$

Comparing (2.11) and (1.19), we have at the point M

$$\Psi\,(1 + \Psi_1^2 + \Psi_2^2) = 0 \tag{2.12}$$

A contradiction is obtained, whence it follows that the surfaces cannot be tangential at the point M. Suppose that Equations

$$u_1 = u_1(\xi), \qquad u_2 = u_2(\xi), \quad \Psi = \Psi\,(u_1(\xi), u_2(\xi)) \tag{2.13}$$

determine an arbitrary curve σ, lying on the surface (1.4), such that with increase in the value of the parameter ξ the curve passes through the point M, approaching the surface of the sphere (1.7) from within.

The vector directed along the tangent to the curve σ at the point M will be written in the form $\tau_0 = (u_1', u_2', \psi')$, where primes denote differentiation with respect to ξ and the point M corresponds to the value of the parameter ξ.

Since the surfaces (1.4) and (1.7) do not touch, then at the point M

$$\mathbf{n}_s \cdot \tau_\sigma > 0 \tag{2.14}$$

Along the curve σ we have $R = R\,(u_1, u_2, \Psi) = R\,(\xi)$. We shall show that at the point M when condition (2.14) is fulfilled the following inequality is valid:

$$dR\,/\,d\xi < 0 \tag{2.15}$$

Accordingly, since the point M is arbitrary, we have established hyperbolic character of the system of equations (1.1) in the neighborhood of the curve l on the surface (1.4). The inequality (2.14) in expanded form can be written

$$u_1u_1' + u_2u_2' + \Psi\Psi' > 0 \qquad (\Psi' = \Psi_1u_1' + \Psi_1u_2') \tag{2.16}$$

With the aid of the relation (2.4) this can be reduced to the form

$$\frac{u_2' - u_1'f'}{f - f'u_1} > 0 \tag{2.17}$$

The sign of R coincides with the sign of the expression in braces in (2.2), which can be put in the form

$$P = \varkappa^2\,(\theta_1^2 + \theta_2^2) + \varkappa^2\,(\theta_2\Psi_1 - \theta_1\Psi_2)^2 - \theta\,(1 + \Psi_1^2 + \Psi_2^2) \tag{2.18}$$

For θ_1 and θ_2 along the curve $u_2 = f(u_1)$ from (2.3), (1.3) and (1.7) we obtain

$$\theta_2 = -\frac{1}{\varkappa} f' \frac{L - A^2}{f - f'u_1} \qquad \theta_2 = \frac{1}{\varkappa}\,\frac{L - A^2}{f - f'u_1} \tag{2.19}$$

Taking the total differential of P with respect to ξ we have

$$\frac{dP}{d\xi} = P_1u_2' + P_2u_2' \tag{2.20}$$

$$P_i = 2\varkappa^2 (\theta_1\theta_{1i} + \theta_2\theta_{2i}) + 2\varkappa^2 (\theta_2\Psi_1 - \theta_1\Psi_2) \ (\theta_{2i}\Psi_1 + \theta_2\Psi_{1i} - \tag{2.21}$$
$$- \theta_{1i}\Psi_2 - \theta_1\Psi_{2i}) - \theta_i (1 + \Psi_1^2 + \Psi_2^2) - 2\theta (\Psi_1\Psi_{1i} + \Psi_2\Psi_{2i}) \qquad (i = 1, 2)$$

Reducing the coefficients R_{ik} to the form

$$R_{ik} = Q (-1)^{i+k} u_m u_n \qquad (m \neq i, \ n \neq k; \ m, n, i, k = 1, 2) \tag{2.22}$$
$$Q = \frac{\theta}{\Psi^2 (f - f'u_1)^2} [\Psi^2 (1 + f'^2) + (ff' + u_1)^2]$$

with the aid of (2.3), (2.4) and (2.19) and making use of the relation resulting from (1.1) and (1.16) along $u_2 = f(u_1)$, namely

$$u_1\Lambda_{1i} + u_2\Lambda_{2i} = u_1\Psi_{1i} + u_2\Psi_{2i} = 0 \qquad (i = 1, 2) \tag{2.23}$$

by virtue of (2.7) and (2.17) we obtain

$$\frac{dP}{d\xi} = \frac{Q}{\theta} A^2 (L - A^2) \left(2 + \frac{1}{\varkappa}\right) \frac{u_1'f' - u_2'}{f - f'u_1} < 0 \tag{2.24}$$

Since the signs of $dP/d\xi$ and $dR/d\xi$ coicide, we have proved the following theorem.

T h e o r e m 2.1 . If behind the front of a three-dimensional curvilinear normal detonation wave the gas flow belongs to the class of potential double waves, and a certain fixed curve in the hodograph space corresponds to the surface of the front, then for the system of equations describing double waves this curve is a line indicating its parabolic character, whilst behind the surface of the detonation front this system of equations is always of hyperbolic type.

The Cauchy problem for the system (1.1), formulated along the curve $u_2 = f(u_1)$, in the given case is a proper one.

3. Let us consider the case of steady flow in the class of potential double waves, when the shape of the surface of the strong discontinuity remains unchanged. The system describing steady three-dimensional double waves derived in [1] is obtained by setting in (1.1)

$$\Lambda = K\Psi + M \qquad (K = \text{const}, \ M = \text{const}) \tag{3.1}$$

Then the equation for Λ in the system (1.1) can be dropped. Equations (1.5) take the form

$$X_i = x_i + \Psi_i (x_3 - Kt) \tag{3.2}$$

and in the system of coordinates x_1, x_2, x_3' we shall have steady motion, if we set $x_3' = x_3 - Kt$ and in place of u_3 introduce the velocity $u_3' = u_3 - K$

From the Cauchy data (1.19) and (1.20) for ψ and Λ on the curve (1.8) we obtain

$$\Psi = L / K = \text{const} \tag{3.3}$$

and, consequently, the dependence (1.8) has the form

$$u_1^2 + u_2^2 = A^2 - L^2 / K^2 = a^2 = \text{const} \tag{3.4}$$

(we shall assume that $K > D$). Setting $\sqrt{u_1^2 + u_2^2} = r$, the solution of the Cauchy problem for the function Ψ will be sought in the form $\Psi = \Psi(r)$ From the first equation of the system (1.1) we obtain for Ψ the ordinary equation

$$\theta\Psi'' r + \Psi'\theta(1 + \Psi'^2) - \Psi'(r + \Psi\Psi' - k\Psi')^2 = 0 \tag{3.5}$$

$$\theta = \frac{1}{\varkappa}\left(k\Psi + M - \frac{r^2}{2} - \frac{\Psi^2}{2}\right), \qquad M = B^2 - L \tag{3.6}$$

Conditions (1.24) for Equation (3.5) are given by the Cauchy initial data

$$\Psi(a) = L / k, \qquad \Psi'(a) = L / Ka \tag{3.7}$$

Accordingly, after specifying the value of the parameter K, the magnitude of which characterizes the inclination of the straight line generators of the surface of the strong discontinuity to the axis of x_3', the function Ψ is found as the solution of the Cauchy problem (3.7) for Equation (3.5) and is completely determined. In the equation for the function X and in the Cauchy initial data for it (1.18) and (1.25) let us pass to polar coordinates $u_1 = r\cos\varphi, u_2 = r\sin\varphi$; We then obtain

$$r^2 X_{rr} + (1 - s(r))(X_{\varphi\varphi} + rX_r) = 0 \tag{3.8}$$

$$X - rX_r = 0, \quad \Phi\left(X_r\cos\varphi - X_\varphi\frac{\sin\varphi}{r}, \quad X_r\sin\varphi + X_\varphi\frac{\cos\varphi}{r}\right) = 0 \quad \text{for } r = a \tag{3.9}$$

where

$$s(r) = \frac{(r + \Psi\Psi' - K\Psi')^2 - \theta\Psi'^2}{\theta} \tag{3.10}$$

and the function Φ is arbitrary (when $x_3' = 0$ can arbitrary specify the shape of the director curve).

In the case of an ordinary shock wave $s(a) < 1$, Equation (3.8) in the neighborhood of the curve $r = a$ is of elliptic type and, generally speaking, the Cauchy problem formulated in (3.9) is improper in the classical sense. The given situation is analogous to the situation arising in the problem of steady flow with shock waves past plane and axisymmetric bodies, when for a given shape of shock wave the contour of the streamlined body is sought.

Let us determine the shape of the streamlined body in the case under consideration. Let the surface of the body correspond in the u_1, u_2 plane to the curve

$$u_2 = \sigma(u_1) \tag{3.11}$$

and the relations (3.2) are the equations of this surface under the conditions (3.11). The condition for absence of flow of gas through the surface has the form

$$\mathbf{n}_\sigma \cdot \mathbf{u} = D_\sigma \tag{3.12}$$

where $\mathbf{n}_\sigma$ is the normal to the surface and D_σ is the normal velocity of motion of the streamlined surface. In exactly the same way as in Section 1, we find that condition (3.12) is equivalent to the two relations

$$(\Psi_{11} + \Psi_{12}\sigma')\,\theta_2 - (\Psi_{21} + \Psi_{22}\sigma')\,\theta_1 = 0 \qquad (3.13)$$

$$(X_{11} + X_{12}\sigma')\,\theta_2 - (X_{21} + X_{22}\sigma')\,\theta_1 = 0 \qquad (3.14)$$

From (3.13), since $\theta = \theta(r)$ and $\Psi = \Psi(r)$, we obtain the result

$$\theta'\Psi' r^{-2}\,(u_2 - u_1\sigma') = 0 \qquad (3.15)$$

Excluding from consideration the trivial case when the streamlined surface is the plane $(u_2 - u_1\sigma' = 0)$, and also the case $\Psi' = 0$, corresponding to a cylindrical surface, we conclude that the equation (3.5) must be integrated, starting with $r = a$ up to such $r = d$ that $\theta'(d) = 0$. From (3.2) it follows that the streamlined surface, just like the surface of the strong discontinuity, is a ruled surface. Conditions (3.14) can be written down in the form

$$\theta' r^{-1}\,[(X_{11} - X_{12}\sigma')\,u_2 - (X_{21} + X_{22}\,\sigma')\,u_1] = 0 \qquad (3.16)$$

and when $\theta'(d) = 0$ it is indeed satisfied, whilst the dependence (3.11) has, accordingly, the form $u_1^2 + u_2^2 = \text{const}$.

We note that when $X \equiv 0$ we obtain the solution of the problem of steady flow past a circular cone. Moreover, if α is the angle included by the cone corresponding to the shock wave, then $\sin \frac{1}{2}\alpha = D/K$, and all the required quantities depend only upon the argument $\sqrt{x_1^2 + x_2^2}/x_3'$ (see [7]). The angle β included by the streamlined cone is found after calculation of the solution of the Cauchy problem (3.7) from Equation (3.5) in the interval $[a, d]$ $d > a$ (the shape of the shock wave is assumed known) from the relation $\tan \frac{1}{2}\beta = \Psi'(d)$.

Accordingly, the function Ψ for the solution of problems of steady flow past three-dimensional bodies in the class of potential double waves can be taken from the corresponding self-similar problem of flow of a uniform supersonic stream past a circular cone. In the determination of the "distribution" function X , however, there remains the degree of arbitrariness noted above.

From the properties of steady self-similar flow past a circular cone it follows that $\Psi'\Psi'' < 0$ when $r \in [a, d]$. Hence, representing $1 - s$ with the aid of (3.5) and (3.10) in the form

$$1 - s = -\Psi'' r / \Psi' \qquad (3.17)$$

we conclude that $1 - s > 0$ and Equation (3.8) in the whole annulus $r = a$, $r = d$ is of elliptic type.

Let us investigate the singularities which can arise in passing from the hodograph space to physical space. For this purpose, after transforming to the polar coordinates r and φ in relations (3.2), determining the flow in the x_1, x_2, x_3' space, let us calculate the Jacobian $I = \partial(x_1, x_2)/\partial(r, \varphi)$ for fixed x_3'.

Making use of Equation (3.8) we obtain for I the expression

$$I = \frac{\Psi'}{\Psi''}(X_{rr} - \Psi'' x_3')^2 - \frac{1}{r}\left(X_{r\varphi} - \frac{X_\varphi}{r}\right)^2 \tag{3.18}$$

From the inequality $\Psi'\Psi'' < 0$ we conclude that $I \leqslant 0$ and I vanishes only under conditions

$$X_{rr} - \Psi'' x_3' = 0, \qquad X_{r\varphi} - r^{-1} X_\varphi = 0 \tag{3.19}$$

If the second equation (3.19) corresponds to certain curves $r = r(\varphi)$ in the annulus under consideration, then the relations (3.2) and (3.19) determine, generally speaking, certain limit lines which can, in this way, appear in the flows under study.

Specifying initial conditions (3.9) not when $x_3' = 0$, as has happened until now, but with sufficiently large values of $|x_3'| = N$, from Expression (3.18) for I and from the fact that for flow past a circular cone

$$I = \Psi'\Psi'' x_3',$$

we find that the region of definition of the solution in physical space in one of the directions along the x_3'-axis is unbounded.

The equations of the director curve in this case, if the form of the conditions in (3.9) is retained, will have the form

$$\begin{aligned} x_1 &= X_r \cos\varphi - X_\varphi r^{-1} \sin\varphi - \Psi' N \cos\varphi \\ x_2 &= X_r \sin\varphi + X_\varphi r^{-1} \cos\varphi - \Psi' N \sin\varphi \end{aligned} \quad \text{for } r = a \tag{3.20}$$

i.e. the shape of the section of the shock wave is close to a circle, and the flow is completely defined in the whole region between the shock wave and the streamlined body for $|x_3'| \geqslant N$.

The simplest examples of flow of the type under consideration are easily obtained in the following way. In the coefficients of Equation (3.8) the variable φ does not appear, and this permits separation of the variables. Setting $X = F(r)\nabla(\varphi)$, we obtain for F and ∇ the equations

$$r^2 F'' + (1 - s)(\lambda F + rF') = 0, \tag{3.21}$$

$$\nabla'' - \lambda \nabla = 0, \qquad \lambda = \text{const} \tag{3.22}$$

Having taken, for example, the solution of Equation (3.8) in the form $X = \cos 2\varphi F_2(r)$, where F_2 is determined from Equation (3.21) when $\lambda = -4$ and $aF'(a) - F(a) = 0$, we obtain surfaces of the shock wave and the streamlined body which are symmetric with respect to the planes $x_1 = 0$ and $x_2 = 0$, given by Equations

$$\begin{aligned} x_1 &= F_2' \cos 2\varphi \cos\varphi + 2F_2 \frac{\sin 2\varphi \sin\varphi}{r} - \Psi' \cos\varphi x_3' \quad \text{for } r = a,\ r = d \\ x_2 &= F_2' \cos 2\varphi \sin\varphi - 2F_2 \frac{\sin 2\varphi \cos\varphi}{r} - \Psi' \sin\varphi x_3' \quad \text{for } r = a,\ r = d \end{aligned} \tag{3.23}$$

In the case of specification of initial data when $x_3' = 0$ it is necessary in each actual solution to verify that X_{rr} and $X_{r\varphi} - r^{-1}X_{\varphi}$ do not vanish simultaneously. If both these expressions do not vanish simultaneously, then in the neighborhood of the plane $x_3' = 0$ it is indeed possible to pass from the hodograph space to physical space.

Note 1. Fourier's method of solution of the Cauchy problem for Equation (3.8) can be effectively applied for construction of the flow behind normal detonation waves when Equation (3.8) is of hyperbolic type.

Note 2. When $|x_3'| \to \infty$ for any initial data the shape of the shock wave and the streamlined body approximates to the shape of a circular cone, and the corresponding flow tends to flow past a circular cone.

Note 3. The foregoing considerations allow us to formulate the problem of flow past a body, the surface of which is the envelope of a family of cones with vertices on a certain curve in the plane $x_3' =$ const and such that all the generators of the cones have one and the same angle of inclination to the x_3'-axis. Moreover, apparently, for certain special curves in the plane $x_3' =$ const we can obtain "complete" flow, i.e. without limiting curves. The solution of this problem is connected with finding the solution of the Cauchy problem for the elliptic equation (3.8). Questions of the existence of such solutions are not considered in the present paper. We notice that, instead of the Cauchy problem for Equation (3.8), we can solve the nonlinear mixed problem in a nonsimply connected region with condition (a) (3.9) when $r = a$ and condition (b) (3.9) when $r = d$ (i.e. in the annulus), when for $x_3' =$ const we have specified the shape of the section, not of the shock wave, but of the streamlined body.

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